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Characterisation of Plus-Constructive Fibrations

A. J. BERRICK

Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 0511

For the numerous applications of the plus-construction, a key question concerns when a fibration $F \to E \to B$ induces another $F^+ \to E^+ \to B^+$. A complete solution (with proof) is given, together with a more easily verifiable simplification in special cases.

1. INTRODUCTION AND MAIN THEOREM

Although originally devised to define the higher algebraic K-theory of rings [1, 7, 16], the plus-construction has quickly established its usefulness in such diverse areas as stable homotopy theory [15], bordism of manifolds [10], and the study of knot complements [14].

Recall (from, e.g., [1] whose notation we follow) that the plus-construction q_X: $X \to X^+$ is a pointed cofibration which induces isomorphisms on homology with Abelian local coeficients (equivalently, has acyclic fibre) and an epimorphism on fundamental groups whose kernel is the maximal perfect subgroup $\mathscr{P}\pi_1(X)$ of $\pi_1(X)$. (These perfect radicals play an important role: in particular an epimorphism $G \twoheadrightarrow^{\phi} H$ preserving perfect radicals ($\phi\mathscr{P}G = \mathscr{P}H$) will be said to be $\mathrm{EP}^2\mathrm{R}$ [3].)

Evidently, the cardinal question is the effect of the construction on homotopy groups. (For instance, when $X = BGLA$, A a ring, then $\pi_j X^+ = K_j A$, the jth algebraic K-theory group of A.) Since the principal tool here is the homotopy exact sequence of a (homotopy) fibration, this effectively reduces to asking *when a fibration* $\mathscr{E}$: $F \to E \to^p B$ *is plus-constructive* (*p.-c.*), that is when $F^+ \to E^+ \to B^+$ is also a fibration (i.e., $F^+ \simeq F_{p^+}$). We assume that all spaces discussed are of the homotopy type of a connected CW-complex which (for (iii) below) admits a finite Postnikov decomposition.) To date, the literature contains numerous sufficient conditions (e.g., [2, 6, 8, 9, 13, 17, 18]) and the necessary condition that $\pi_1(p)$ be $\mathrm{EP}^2\mathrm{R}$ [1, (6.8)]. In view of the awesome complication of some of these (sets of) hypotheses, it is remarkable that a simple characterisation of when $\mathscr{E}$ is p.-c. is after all possible. (The brevity of the proof is a further blessing.)

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THEOREM. *The following are equivalent statements about a fibration* $\mathscr{E}$: $F \to E \to^p B$.

(i) $\mathscr{E}$ *is plus-constructive.*

(ii) $\mathscr{P}\pi_1(B)$ *acts on* F^+ *by maps (freely) homotopic to the identity.*

(iii) (a) $\pi_1(p)$ *is* $\mathrm{EP^2R}$, *and*

(b) $\mathscr{P}\pi_1(E)$ *acts trivially on* $\pi_*(F^+, *)$.

Statements (ii) and (iii) above use the naturality of the plus-construction to obtain an action on F^+ (pointed, if appropriate) from a corresponding one on F; hence similarly for homotopy groups. Since we consider actions by *perfect groups*, it is important to bear in mind the result [1, 2] that *such an action on a group is nilpotent if and only if it is trivial.* The proof is presented in Section 3 after the description of some corollaries. A final paragraph looks at some pertinent examples.

2. Consequences and Special Cases

As ever, $\mathscr{E}$: $F \to E \to B$ is a fibration. (Also, $\bar{\ }$ denotes normal closure.)

COROLLARY 1. *Let* f: $B' \to B$ *be a map. If* $\mathscr{E}$ *is plus-constructive (p.-c.) then the induced fibration* $f^*\mathscr{E}$ *is also p.-c. The converse also holds when* $\overline{f_* \mathscr{P}\pi_1(B')} = \mathscr{P}\pi_1(B)$.

This is immediate from the equivalence of (i) and (ii) in Section 1. We know of no direct proof of these results. The following special case of the main theorem seems to take care of all situations so far published:

COROLLARY 2. *Suppose* F^+ *is a nilpotent space. Then* $\mathscr{E}$ *is p.-c. if and only if* $\mathscr{P}\pi_1(B)$ *acts trivially on* $H_*(F; \mathbb{Z})$.

The sufficiency of the homology condition is established in [2]. Alternatively (at least when F^+ is finite-dimensional), one can appeal to [5, Theorem D] which asserts that the image of $\mathscr{P}\pi_1(B)$ in the group $\mathrm{AUT}\, F^+$ of self-equivalences is nilpotent, and hence, being perfect, trivial. So (ii) of Section 1 applies. For the converse, Section 1 (ii) implies the triviality of the composition

$$\mathscr{P}\pi_1(B) \hookrightarrow \pi_1(B) \to \mathrm{AUT}\, F^+ \to \prod_j \mathrm{aut}\, H_j(F^+) \cong \prod_j \mathrm{aut}\, H_j(F).$$

Since many applications concern classifying spaces of groups, the following special case of Section 1 (iii) is worthwhile. (Of course $\mathscr{P}N = 1$ leaves $BN^+ = BN$.) It will be exploited in Section 4.

COROLLARY 3. *Suppose* $N \hookrightarrow G \twoheadrightarrow^{\phi} Q$ *is a group extension, with* $\mathscr{P}N = 1$. *Then the fibration* $BN \to BG \to BQ$ *is p.-c. if and only if both* ϕ *is* $\mathrm{EP^2R}$ *and* $[N, \mathscr{P}G] = 1$.

3. PROOF OF THE THEOREM

The idea of the proof is quite simple: it consists of passing to another fibration where p.-c. can be shown equivalent to fibre homotopy trivial and thus to the nulhomotopy of the classifying map from the base to the universal fibration. This passage comprises two steps. The first, performed in [2], is the *fibre-wise plus-construction.* In terms of universal fibrations (see [5, Sect. 4] for notation), it amounts to the observation that the naturality of the plus-construction gives rise to a map

$$\begin{array}{ccc} F & \xrightarrow{q_F} & F^+ \\ \downarrow & & \downarrow \\ B\,\mathrm{aut}^0 F & \longrightarrow & B\,\mathrm{aut}^0 F^+ \\ \downarrow & & \downarrow \\ B\,\mathrm{aut}\, F & \xrightarrow{f} & B\,\mathrm{aut}\, F^+ \end{array}$$

from $\mathscr{F}$ to $\mathscr{F}^+$, say. Then the fibre-wise plus-construction on $\mathscr{E} = g^*\mathscr{F}$ (say) is the fibration $(f \circ g)^*\mathscr{F}^+$ with base B, fibre F^+; because the map on fibres from $\mathscr{E}$ to $(f \circ g)^*\mathscr{F}^+$ is acyclic, so too must be the map of total spaces [1, (4.2)], making $\mathscr{E}$ p.-c. precisely when $(f \circ g)^*\mathscr{F}^+$ is. The second step uses the lemma [2, (2.1)] which implies that for any fibration $A \to B \to C$ with $\mathscr{P}\pi_1(C) = 1$, a fibration over B is p.-c. if and only if its pull-back over A is. So we pull back $(f \circ g)^*\mathscr{F}^+$ over $AB \to^h B$, the acyclic fibre of $q_B \colon B \to B^+$ [1, (7.7)], obtaining $(f \circ g \circ h)^*\mathscr{F}^+ \colon F^+ \to E_1 \to AB$. This is p.-c. just so long as there is a map of fibrations

$$\begin{array}{ccc} F^+ & \xrightarrow{id} & F^+ \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{q_{E_1}} & E_1^+ \\ \downarrow & & \downarrow \\ AB & \longrightarrow & * \end{array}$$

that is, F^+ is a retract of E_1. But by the Dold theorem [4], this is equivalent (cf. [11, (4.3)] to the fibre homotopy triviality of $(f \circ g \circ h)^* \mathscr{F}^+$, or equally, the nullhomotopy of $AB \to^h B \to^{f \circ g} B \operatorname{aut} F^+$. However, because B^+ is also the cofibre of h [1, (7.7)], this corresponds to the existence of a factorisation of $f \circ g$ through q_B: $B \to B^+$; a universal property of q_B [1, (5.2)] reduces this to the triviality of $\pi_1(f \circ g)$ on $\mathscr{P}\pi_1(B)$, which is a restatement of (ii).

For (iii), we argue from the commuting diagram of group homomorphisms

$$\begin{array}{ccccc} & & \pi_1 F^+/(K = \operatorname{Im} \pi_2 B \operatorname{aut} F^+) & & \\ & & \downarrow & & \\ \mathscr{P}\pi_1 E & \xrightarrow{\beta} & \mathrm{AUT}^0 F^+ & \xrightarrow{\delta} & \prod \operatorname{aut} \pi_j(F^+, *) \\ {\scriptstyle\alpha}\downarrow & & \downarrow & & \\ \mathscr{P}\pi_1 B & \xrightarrow{\gamma} & \mathrm{AUT}\, F^+ & & \end{array}$$

whose right-hand sequence is exact. Condition (ii) is that γ is trivial. So $\operatorname{Im} \beta$ (perfect) lifts to $\mathscr{P}(\pi_1 F^+/K)$, a factor group of $\mathscr{P}\pi_1 F^+$ ([3, (3.1)], K being central) and therefore trivial. A similar sort of argument [3, (2.2)] shows that α must be onto. Conversely, suppose $\operatorname{Im} \beta$ lies in $\operatorname{Ker} \delta$: the former subgroup is again perfect, while the latter is, by [5, Theorem B], nilpotent. This again makes β, hence $\gamma \circ \alpha$, trivial. When α is surjective this forces γ to be trivial, once more clinching the equivalence to statement (ii).

4. Examples

We consider two general classes of example, both classifying space fibrations, in order to demonstrate that where hypotheses occur in the above in pairs, neither premise in the pair is superfluous.

The easiest example of a non-EP^{2}R epimorphism, and hence non-p.-c. classifying space fibration, arises from a presentation $R \hookrightarrow F \twoheadrightarrow Q$ of the image Q, since the free group F has $\mathscr{P}F = 1$. However $[R, \mathscr{P}F] = 1$ certainly holds for Corollary 3 and so Section 1(iii).

Secondly, we recall from [1, Chap. 3] that for any ring A there is a split extension $\mathrm{MA} \rightarrowtail^{\iota} \mathrm{GLUT} \twoheadrightarrow^{\phi} \mathrm{GL}(A \oplus A)$. Here MA is the group under addition of all finite matrices, while GLUT denotes the general linear group on the ring of upper triangular 2×2 matrices over A. On the one hand, any split epimorphism is EP^{2}R [3]. On the other, the matrices $I + (\delta_{1r}\delta_{2s})_{r,s}$ and $I + (\delta_{2r}\delta_{3s})_{r,s}$, in $\mathscr{P}\mathrm{GLUT}$, $\iota\mathrm{MA}$, respectively, fail to commute. Thus by Corollary 3 the fibration $B\mathrm{MA} \to B\mathrm{GLUT} \to B\mathrm{GL}(A \oplus A)$, despite having a section, cannot be p.-c. (A less direct proof appears in [1].) Moreover, since

MA, hence BMA, is nilpotent, the action of $\mathscr{P}$GL($A \oplus A$) ($=E(A \oplus A)$) on $H_*(B\text{MA})$ cannot be trivial (Corollary 2). Application of Section 1 (ii) springs a surprise here, for the section factorises $\mathscr{P}\text{GL}(A \oplus A) \to \text{AUT}(B\text{MA})$ as $\mathscr{P}\text{GL}(A \oplus A) \to \text{GLUT} \to \text{AUT}^0(B\text{MA}) \to \text{AUT}(B\text{MA})$. Hence the group $\text{AUT}^0(B\text{MA})$ of pointed self-equivalences of BMA is nonsolvable, admitting a nontrivial perfect subgroup. It is noteworthy to compare this with Whitehead's theorem [12, 19], to the effect that when X is a group-like space of *finite* category, then $\text{AUT}^0 X$ is not merely solvable but nilpotent.

REFERENCES

1. A. J. BERRICK, "An Approach to Algebraic K-Theory," Research Notes in Math. No. 56, Pitman, London, 1982.
2. A. J. BERRICK, The plus-construction and fibrations, *Quart. J. Math. Oxford Ser.* (2) **33** (1982), 149–157.
3. A. J. BERRICK, Epimorphisms preserving perfect radicals, *J. Australian Math. Soc.*, to appear.
4. A. DOLD, Partitions of unity in the theory of fibrations, *Ann. of Math.* **78** (1963), 223–255.
5. E. DROR AND A. ZABRODSKY, Unipotency and nilpotency in homotopy equivalences, *Topology* **18** (1979), 187–197.
6. S. M. GERSTEN, On the spectrum of algebraic K-theory, *Bull. Amer. Math. Soc.* **78** (1972), 216–219.
7. S. M. GERSTEN, Higher K-theory of rings, *in* "Lecture Notes in Mathematics" No. 341, pp. 1–40, Springer–Verlag, Berlin, 1973.
8. S. M. GERSTEN, K_3 of a ring is H_3 of the Steinberg group, *Proc. Amer. Math. Soc.* **37** (1973), 366–368.
9. J.-C. HAUSMANN, Manifolds with a given homology and fundamental group, *Comment. Math. Helv.* **53** (1978), 113–134.
10. J.-C. HAUSMANN AND P. VOGEL, The plus construction and lifting maps from manifolds, *Proc . Symp. Pure. Math.* **32** (1978), 67–76.
11. I. M. JAMES, "The Topology of Stiefel Manifolds," London Math. Soc. Lecture Notes No. 24, Cambridge Univ. Press, London/New York, 1976.
12. I. M. JAMES, On category, in the sense of Lusternik–Schnirelmann, *Topology* **17** (1978), 331–348.
13. J. P. MAY, E_∞ spaces, group completions, and permutative categories, *in* London Math. Soc. Lecture Notes, No. 11, pp. 61–93, Cambridge Univ. Press, London/New York, 1974.
14. W. MEIER, Acyclic maps and knot complements, *Math. Ann.* **243** (1979), 247–259.
15. S. B. PRIDDY, Transfer, symmetric groups, and stable homotopy-theory, *in* Lecture Notes in Mathematics No. 341, Springer–Verlag, Berlin, 1973.
16. D. QUILLEN, Cohomology of groups, *in* "Actes, Congrès intern. Math. 1970," Tome 2, pp. 47–51, Gauthier–Villars, Paris, 1971.
17. J. B. WAGONER, Delooping classifying spaces in algebraic K-theory, *Topology* **11** (1972), 349–370.
18. C. A. WEIBEL, K-theory of Azumaya algebras, *Proc. Amer. Math. Soc.* **81** (1981), 1–7.
19. G. W. WHITEHEAD, On mappings into group-like spaces, *Comment. Math. Helv.* **28** (1954), 320–328.